Confidence intervals for rank order statistics: Somers' D, Kendall's τ_a and their differences

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- Somers' D and Kendall's τ_a
- Why rank order statistics?
- Why confidence intervals?
- The program `somersd`
- An example (from the `auto` data)

Parameters behind "non-parametric" statistics

Suppose (X_1, Y_1) and (X_2, Y_2) are sampled independently from the same bivariate population. Kendall's τ_a (the signed difference covariance) is

$$\tau_{XY} = E\big[\,\mathrm{sign}(X_1 - X_2)\,\mathrm{sign}(Y_1 - Y_2)\,\big],$$

and is the difference between the two probabilities of concordance and discordance.

Somers' D (the corresponding regression coefficient) is

$$D_{YX} = \tau_{XY}/\tau_{XX},$$

and is the difference between the corresponding *conditional* probabilities, given unequal X-values.

These parameters are estimated by corresponding sample statistics, which are used to test null hypotheses. If X is binary, then `ranksum` tests $H_0 : D_{YX} = 0$.

Why rank order statistics?

No statistical method is simply "robust". However, Kendall's τ_a and Somers' D are robust to:

- Extreme values. (These often "throw" classical regression and correlation coefficients.)
- Non-linearity. (They are not affected by monotonic transformations, and can be ± 1 for perfect non-linear relationships.)

They are useful as a preliminary to homing in on a particular regression model. (And losing a section of the skeptical public, who do not believe the model assumptions.)

Why confidence intervals?

- They *might* discourage people from arguing that a high P-value proves a null hypothesis.
- For continuous data, we often have Greiner's correspondence between Kendall's τ_a and Pearson's ρ,

  $$\rho = \sin\left(\frac{\pi}{2}\,\tau\right).$$

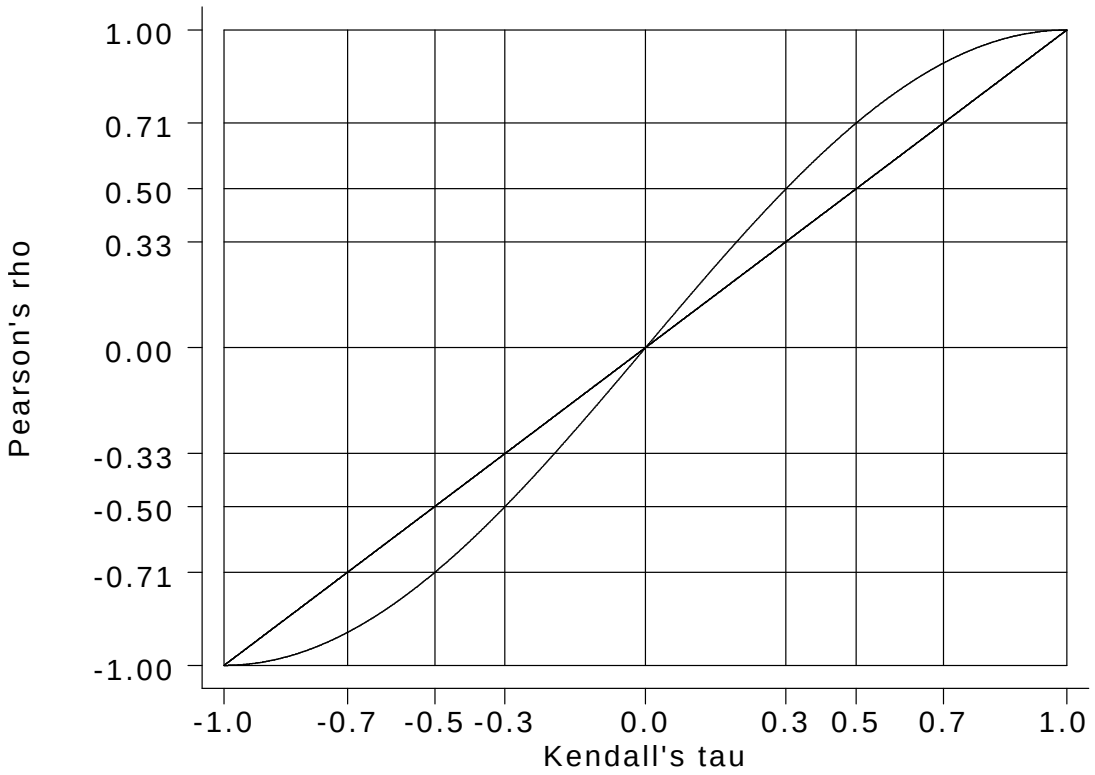
  So, given a CI for τ, we can define an "outlier-resistant" CI for ρ.
- We might want to know CIs for differences, such as

  $$\tau_{XY} - \tau_{WY} \quad \text{or} \quad D_{YX} - D_{YW},$$

  where Y is an outcome variable and W and X are competing predictor variables. This is because a larger τ_a cannot be secondary to a smaller τ_a.

Greiner's relation: $\rho = \sin\left(\frac{\pi}{2}\tau\right)$

- This holds under the bivariate normal distribution, and many other continuous distributions.
- It therefore saves the labour of hunting for a pair of transformations.
- Kendall's τ is less vulnerable than Pearson's ρ to "outliers".
- *However*, τ is "less impressive". ($\tau = \pm 1/3$ when $\rho = \pm 1/2$, and $\tau = \pm 1/2$ when $\rho = \pm 1/\sqrt{2}$.)



somersd - A program to calculate confidence intervals for rank order statistics

- Calculates jackknife confidence intervals for either Somers' D or Kendall's τ_a, between one variable X and a list of others $Y^{(1)} \ldots Y^{(p)}$.
- Offers a choice of transformations: Fisher's z, Daniels' arcsine, Greiner's ρ, and the z-transform of Greiner's ρ.
- A `cluster()` option is available (mostly for measuring intra-class correlation, eg between twin sisters).
- Estimation results are saved as for a model fit, so differences can be estimated using `lincom`.

Example: Weight and fuel consumption in US and non-US cars (1)

In the `auto` data set, we use `somersd` (instead of `ranksum`) to assess US origin as a predictor of fuel consumption (gallons/mile) and weight (lbs.). We use Fisher's z-transform to compute asymmetric confidence limits for the two Somers' Ds:

```
. somersd us gpm weight,tran(z)
Somers' D
Transformation: Fisher's z
Valid observations: 74
------------------------------------------------------------------------------
             |              Jackknife
          us |      Coef.   Std. Err.       z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gpm |   .4937249   .1708551     2.890   0.004     .1588551    .8285947
      weight |   .9749561   .1908547     5.108   0.000     .6008878    1.349024
------------------------------------------------------------------------------
95% CI for untransformed Somers' D
            Somers_D    Minimum    Maximum
       gpm  .45716783  .15753219  .67972072
weight  .75087413  .53768098  .87382282
```

We note that US cars (usually) are heavier than the rest, and consume more fuel per mile.

Example: Weight and fuel consumption in US and non-US cars (2)

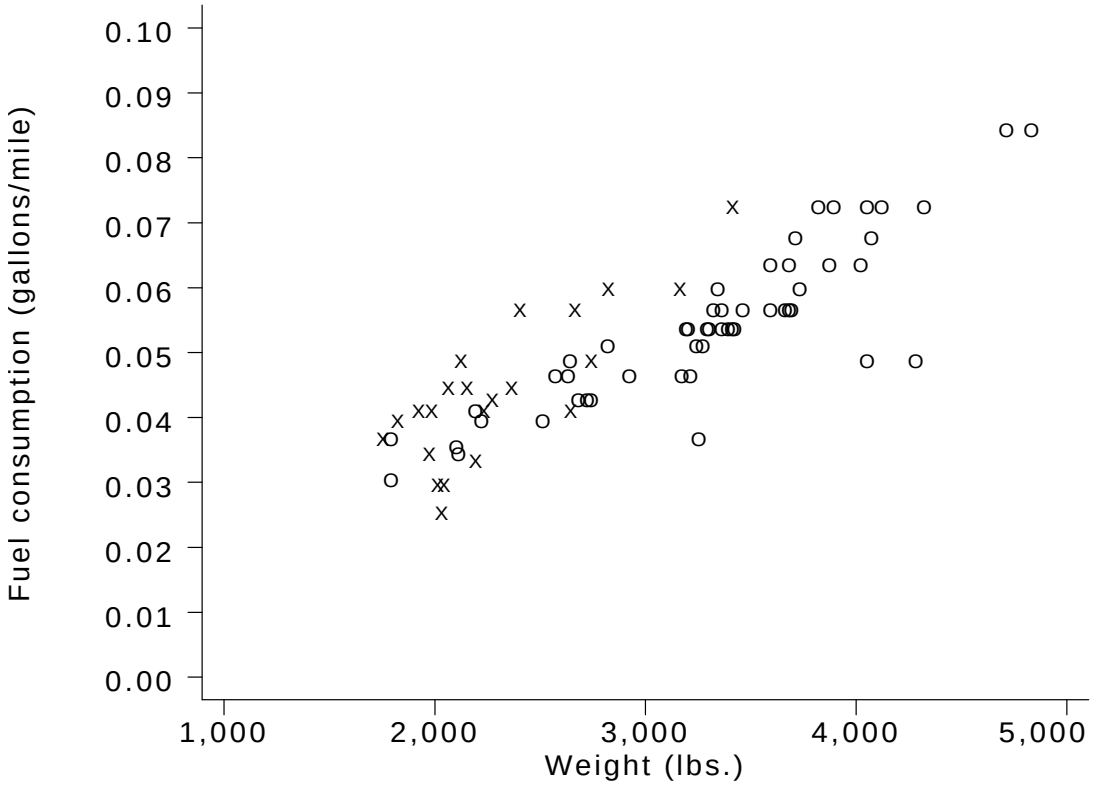
Using `lincom`, we can show that US origin predicts weight better than it predicts fuel consumption. We estimate the difference (in z-units) between the two positive z-transformed Somers' D values:

```
. lincom weight-gpm
 ( 1) - gpm + weight = 0.0
------------------------------------------------------------------------------
      us |      Coef.   Std. Err.       z     P>|z|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     (1) |   .4812312   .1235452      3.895   0.000       .2390871    .7233753
------------------------------------------------------------------------------
```

The difference is positive. So, given two cars, one consuming fewer gallons to move more weight, the other consuming more gallons to move less weight, the *first* (the *more efficient* one) is more likely to be US-made.

Example: Weight and fuel consumption in US and non-US cars (3)

- Data points are US models ("O") and non-US models ("X").
- Given two models, one moving more mass with less gas, the other moving less mass with more gas, the *first* is more likely to be a US model.
- Using `somersd`, we can show this without contentious assumptions such as linearity, because the difference between the two Somers' Ds depends entirely on such disconcordant pairs.



Further reading

Newson, R. 2000. `somersd` - Confidence intervals for "non-parametric" statistics and their differences. *Stata Technical Bulletin* 55, in press. (To appear in May 2000.)

Edwardes, M. D. deB. 1995. A Confidence Interval for $\Pr(X < Y)$ - $\Pr(X > Y)$ Estimated From Simple Cluster Samples. *Biometrics* 51: 571-578.

Kendall, M. G. and J. D. Gibbons. 1990. *Rank correlation methods. 5th ed.* New York: Oxford University Press.